ELASTOPLASTIC DEFORMATION AND CONTACT MODELLING USING A MESHLESS METHOD

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Résumé :

La modélisation numérique de manière efficace de problèmes incluant des non linéarités liés au matériau, à la géométrie et aux conditions de contact reste de nos jours un défi. Souvent ce genre de problème est modélisé par des méthodes adaptatives par éléments finis. Les méthodes sans maillage offre des possibilités attractive d'adaptation sans remaillage.la procédure se fait par Une simple addition ou suppression de nœuds. Ce papier cherche à proposer un algorithme de résolution de problème élastoplastic incluant le contact par la méthode « Element Free de Galerkin » basée sur l'approximation des moindres carrés mobiles. Les conditions de contact sont traitées par une méthode de pénalisation. Les multiplicateurs de Lagrange sont utilisés pour corriger les écarts dans les conditions aux limites essentielles. Dans cette étude sont testées une méthode matricielle et la programmation mathématique. Le programme développé est implémenté dans l'environnement Matlab.des exemples pratiques sont présentés à la fin de ce travail.

Abstract :

The numerical modeling problems including both material, geometric and contact nonlinearities remains challenging. Often these problems are modeled with an adaptive finite element method. Meshless methods offer the attractive possibility of simpler adaptive procedures involving no remeshing, simple insertion or deletion of nodes. In this paper, a meshless approach for elastoplastic behavior and contact is developed. The Free Galerkin Method based on Moved Least Squares "MLS" approximation has been used. The penalty method for imposition of contact constraint is proposed and the Lagrange multipliers method is implemented to impose essential boundary conditions. In this work, the mathematical programming and the matrix method are implemented and tested on the Matlab environment.

Mots clefs : élastoplasticité; formulation incrémentale, element free Galerkin, moved least square, pénalité, multiplicateur de Lagrange

Keys words: elastoplasticity, incremental formulation, element free Galerkin, moved least square, penalty, lagrangian multiplier, mathematical programming, matrix method.

1. Introduction

Recently there has been a fast development progress in meshless methods. Particularly, the Element Free Galekin "EFG" method has been applied successfully to non-linear problems in mechanics. We quote the contact **[1]** and plasticity **[2].** The current study is paid attention to performing a meshless model to solve frictionless contact problems with elastoplasticity. The "EFG" method based on *Moved Least Squares* "MLS" approximation is chosen for its performances in stability and convergence. In this paper, the MLS approximation, the elastoplastic model and the contact law will be reviewed. The difficulties due to the imposition of essential boundary conditions and the numerical integration are discussed. To show the feasibility of the model, some practical examples will be presented in the end.

2. MLS approximation

The MLS as an approximation method has been introduced by Shepard **[3]** and Lancaster **[4]**. It consists of three components: a basis function, a weight function associated with each node, and a set of coefficients that depends on node position.

Using MLS approximation, a data value $u = \{u_i\}_{i=1}^{N}$ at nodes x_i is approximated by a function $u^h \in C^s(\mathbb{R}^d)$ in a weighted square sense namely [1,5]:

$$u^h(x) = \sum_{i=1}^{m} p_i(x) a_i(x) = p^T(x) a(x) \tag{1}$$

Where $p_i(x), i = 1,2, \dots .m$ are monomial basis functions, $p^T(x) = [p_1(x), p_1(x), \dots, p_m(x)]$, m is the number of terms in the basis, and $a_i(x)$ are the coefficients of the basis functions. In general, the basis functions are as follows: For example, for a 1-D problem, the linear basis is: $\{1, x\}$, and the quadratic basis is $\{1, x, x^2\}$. For a 2-D problem, the linear basis is: $\{1, x, y\}$, and the quadratic basis is $\{1, x, y, x^2, xy, y^2\}$ where $x = (x, y)$.

The coefficient vector $a(x)$ is determined by minimizing a weight discrete square norm, which is defined as

$$J(x) = \sum_{i=1}^{n} w_i(x) \left(p^T(x) a(x) - u_i\right)^2 \tag{2}$$

where $w_i(x)$ is the weight function associated with the node *i*, *n* is the number of nodes in Ω for which the weight function $w_i(x) > 0$ and u_i are the fictitious nodal values, but not the nodal values of the unknown trial function $u^h(x)$ i.e. $u^h(x_i) \neq u_i$.

The stationary point of *J*, in equation (2), with respect to $a(x)$ leads to the following linear relation between $a(x)$ and u:

$$A(x) a(x) = B(x) u \tag{3}$$

where the matrices $A(x)$ and $B(x)$ are defined by:

$$A(x) = P^T W P = B(x) P = \sum_{i=1}^{n} w_i(x) p(x_i) p^T(x_i) \tag{4}$$

$$B(x) = P^T W = [w_1(x) p(x_1), w_2(x) p(x_2), \dots, w_n(x) p(x_n)] \tag{5}$$

The matrix A is often called the moment matrix, it is of size $(m+1) \times (m+1)$.Computing $a(x)$ using equation (3) and substituting it into equation (1), give:

$$u^h(x) = \Phi^T(x).u = \sum_{i=1}^{n} \phi_i(x).u_i\ , \quad x \in \Omega \tag{6}$$

Where

$$\Phi^T(x) = p^T(x) A^{-1}(x) B(x) \tag{7}$$

Or

$$\phi_i(x) = \sum_{k=1}^{m} p_k(x) [A^{-1}(x) B(x)]_{ki} \tag{8}$$

$\phi_i(x)$ are called the shape functions of the MLS approximation, corresponding to nodal point x_i, similar to the interpolation function in FEM .

3. Elastoplastic analysis

The classical hypothesis of elastoplastic decomposition of the total strain rate in two parts is considered **[6,7]:**

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p . \tag{9}$$

Where $\dot{\varepsilon}^e$ is the elastic part verifying the generalised Hooke's law and $\dot{\varepsilon}^p$ the plastic part given by the classical normality law.

The plastic flow is a quasistatic problem. Consequently, the time plays the role of a simple parameter of evolution. The stress state doesn't depend on the intensity of the velocity. While applying the implicit integration scheme introduced first by Moreau **[8]** and known as the "catching up" algorithm, the time can be eliminated. This method suggests calculating of a function called the superpotential of dissipation. For the incremental formulation, the following notations will be used:

$$\begin{aligned} \Delta\varepsilon^p &= \varepsilon_1^p - \varepsilon_0^p \quad , \quad \Delta\varepsilon^e = \varepsilon_1^e - \varepsilon_0^e , \\ \Delta\sigma &= \sigma_1 - \sigma_0 \quad , \quad \Delta\varepsilon = \varepsilon_1 - \varepsilon_0 , \end{aligned} \tag{10}$$

where the index 0 (resp. 1) is relative to the beginning (resp. to the end) of the step.
According to the implicit integration, the plastic strain increment is given by [6,7]:

$$\Delta\varepsilon^p = \Delta\tau\dot{\varepsilon}_1^p . \tag{11}$$

In the frame of Convex Analysis, the incremental law and its inverse will be expressed by the following relations [6,7]:

$$\Delta\varepsilon^p \in \partial_{\Delta\sigma}\Delta W_p(\Delta\sigma)\,;\; \Delta\sigma \in \partial_{\Delta\varepsilon^p}\Delta V_p(\Delta\varepsilon^p) , \tag{12}$$

where ΔV_p represents the incremental energy of plastic deformation and ΔW_p is its complementary. ΔV_p can be calculated from the so-called plastic dissipation which, for the Von-Mises material, is given by the following expression:

$$V_p(\dot{e}^p) = \sqrt{\tfrac{2}{3}}\ \sigma_Y \left\| \dot{e}^p \right\| , \tag{13}$$

where σ_Y is the yield stress of the material and the symbol $\left\| \dot{e}^p \right\|$ represents the Euclidean norm of the plastic strain deviator $\dot{e}^p$.

Finally, to determine the incremental elastoplastic superpotentiel $\Delta V(\Delta\varepsilon)$ that will be used in the variational formulation, the inf-convolution concept is used and defined by **[6,7]:**

$$\begin{aligned} \Delta V &= \Delta V_e \otimes_{\Delta\varepsilon^p} \Delta V_p \\ &= \underset{\Delta\varepsilon^p \text{ incompressible}}{Inf} \left\{ \Delta V_e(\Delta\varepsilon^e) + \Delta V_p(\Delta\varepsilon^p) \right\} , \end{aligned} \tag{14}$$

where ΔV_e and ΔV_p are respectively the elastic and plastic incremental superpotentials.

4. Problem of contact

4.1 Contact law

The Signorini law of contact can be rewritten analytically as follows:

If	$t_n = 0$	then	$\dot{u}_n > 0$,	**non contact**	
Else if	$\dot{u}_n = 0$	then	$t_n > 0$,	**contact**	(15)

$\dot{u}_n$ and t_n denote the normal components of velocity and contact force.

4.2 Incremental formulation and penalty procedure

Let's note by 0 and 1 the beginning and the end of the step, the incremental displacement is defined by:

$$\Delta u_n = u_{n1} - u_{n0} = \Delta\tau\dot{u}_{n1} . \tag{16}$$

To implement the penalty method, practically we introduce a fictitious rigidity as follows [6,7]:

$$\Delta u_n = \Delta u_n^f + \frac{\Delta t_n}{\alpha} \tag{17}$$

where α is the penalisation factor. Δu_n^f is fictitious increment computed from the actual displacement increment Δu and the previous of the contact forces increments. The regularisation of the contact law leads us to introduce the following differentiable function, **[6,7]:**

$$\Delta b_c = \alpha(-\Delta u_n + \Delta u_n^f)^2 \tag{18}$$

The detailed procedure of penalization applied to contact problem with the meshless method is presented in [1].

5. Meshless formulation

The approximation of the increment of displacement, can be presented as follow:

$$\Delta u(x) = \Phi^T(x)\Delta U \quad , \quad \Delta\varepsilon = B(x)\Delta U \tag{19}$$

where $\Phi(x)$ is the matrix of the shape functions of the MLS approximation defined by (7,8), corresponding to nodal point x_i and $B(x) = grad_s(\Phi(x))$.

After this step the variational principle concept can be used similarly to "FEM".

Using the variational principle approach, the functional, in the meshless method context is given by:

$$\Delta\Phi(\Delta U) = \int_\Omega \Delta V(B\Delta U)\, d\Omega + \int_{S_2} \Delta b_c(-\Phi^T\Delta U, \Delta t)\, dS - \int_{S_1} \Delta\bar{t}.\Phi^T\Delta U\ dS \tag{20}$$

To solve the system (20), the matrix method and the mathematical programming are used for elastostatic problems. For the nonlinear problems, a procedure of minimisation by the Newton method is used.

6. Difficulties

6.1Boundary condition

The enforcement of essential boundary conditions remains an area of ongoing research, since the shape functions in meshless methods are not strict interpolates, i.e. they do not satisfy the Kronecker delta condition:

$$\phi_i(x_j) \neq \delta_{ij} = \begin{cases} 1 & if\ i = j \\ 0 & othrwise \end{cases}$$

Consequently the approximated value on the boundary depends on interior nodes as well as boundary nodes. Boundary conditions are in terms of linear combinations of nodal values. Various ways of implementing boundary conditions for meshless methods have been suggested, such as combining the Element Free Galerkin method (EFG) with finite element shape functions close to the boundary, the use of a modified variational principle, the use of window or correction functions that vanish on the boundary , and the use of Lagrange multipliers. In this paper, the essential boundary condition is imposed using the Lagrange multipliers method **[1,5].**

6.2 Numerical integration

In the MLS method, the concept of element does not exist, and the shape functions are not polynomial, thus the integrals cannot be evaluated as for the finite element method. However, direct integration nodes or an underlying grid that serves only in the numerical integration, and does not interfere in the approximation scheme can be used. In this way, the second method [1,5] was used.

7. Examples

7.1 hollow disk

a hollow disk is considered as shown (Figure 1). $P = 100\ MPa$, $\nu = 0.3$, $E = 2{,}1.10^5 MPa$, $a = 20\ mm$ and $b = 90\ mm$.

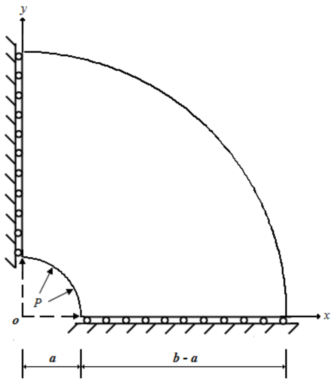

Figure 1 Geometry and boundary conditions

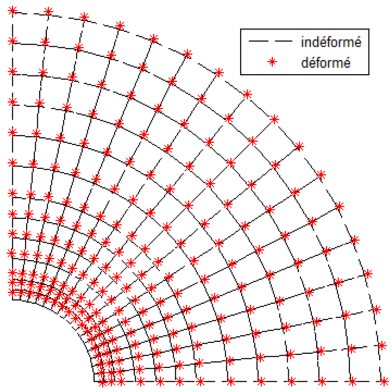

Figure 2 Deformed model

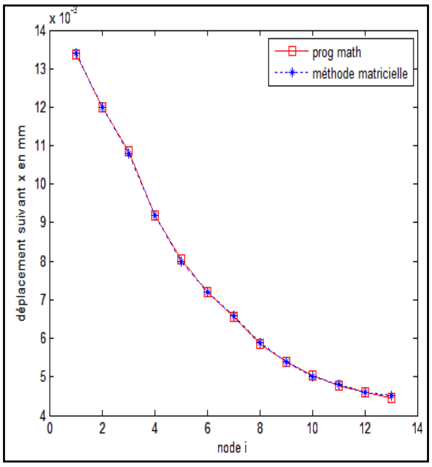

Figure 3 Displacement toward the axis (ox)

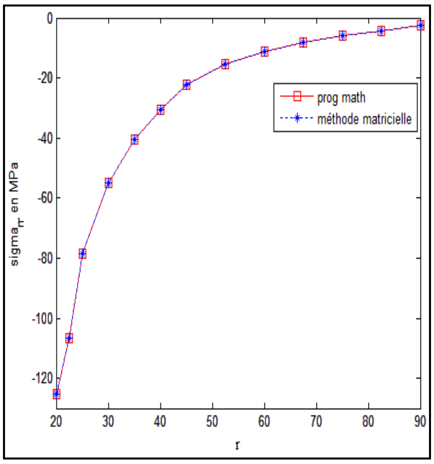

Figure 4 Distribution of σ_{rr} toward r

7.2 **The Contact problem**

A deformable bloc (60mmx30mm) in contact with a rigid foundation is analyzed in this section. The geometry and boundary conditions are showed on Figure 5 and Figure 6 , the mechanical properties are: E= 2.110^4 N/mm^2, υ=0.3.

The surface of contact is defined by the contact nodes. In this work, this surface is considered as known.

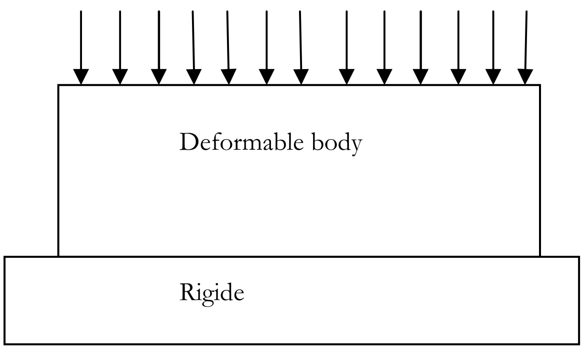

Figure 5 Geometry and load (P=100N)

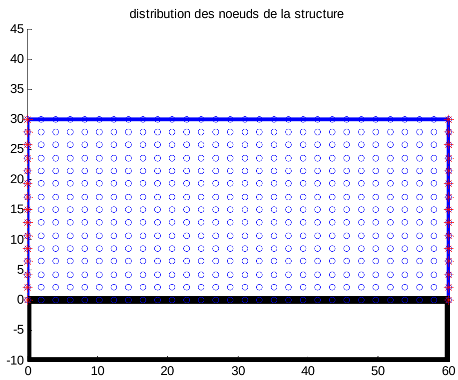

Figure 6 nodal Distribution 30 ×15

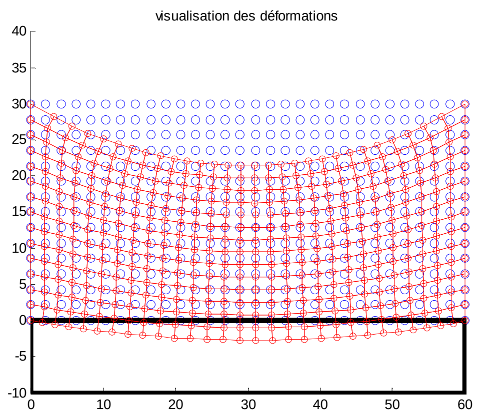

Figure 7 Deformation of the structure for α=10^2

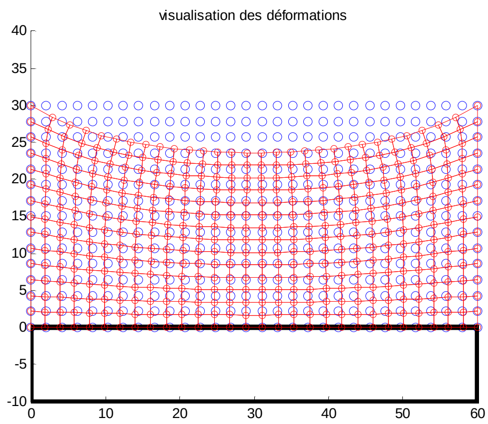

Figure 8 Deformation of the structure for α=10^8

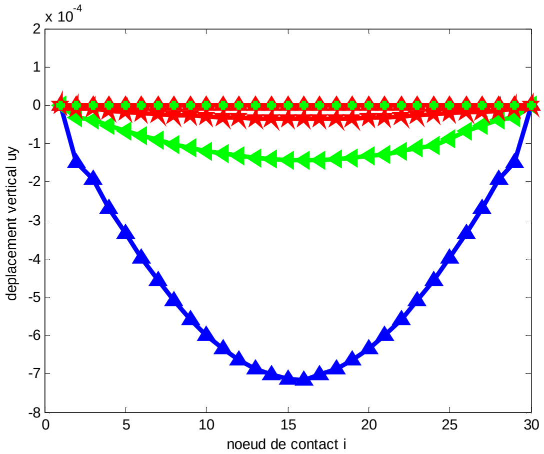

Figure 9 Vertical displacements for different values of the contact penalty parameter (α).

8. Conclusion

An algorithm for solving the elastoplastic evolution problem taking account the unilateral contact using a meshless method has been presented. To show the feasibility of the method, some numerical examples have been presented. The works are under the way for testing the method on complex non-linear problems.

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